Continuous limit of the moments system for the globally coupled phase oscillators

Faculty of Mathematics, Kyushu University, Fukuoka, 819-0395, Japan
Hayato CHIBA [1]

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Abstract

The Kuramoto model, which describes synchronization phenomena, is a system of ordinary differential equations on N-torus defined as coupled harmonic oscillators. The order parameter is often used to measure the degree of synchronization. In this paper, a few properties of the continuous model for the Kuramoto model are investigated. In particular, the moments systems are introduced for both of the Kuramoto model and its continuous model. By using them, it is proved that the order parameter of the N-dimensional Kuramoto model converges to that of the continuous model as $N \to \infty$.

1 Introduction

Collective synchronization phenomena are observed in a variety of areas such as chemical reactions, engineering circuits and biological populations [18]. In order to investigate such a phenomenon, Kuramoto [11, 12] proposed the system of ordinary differential equations

$$\frac{d\theta_i}{dt} = \omega_i + \frac{K}{N}\sum_{j=1}^{N}\sin(\theta_j - \theta_i),\ \ i = 1, \cdots, N, \tag{1.1}$$

where $\theta_i \in [0, 2\pi)$ denotes the phase of an i-th oscillator on a circle, $\omega_i \in \mathbf{R}$ denotes its natural frequency, $K > 0$ is the coupling strength, and where N is the number of oscillators. Eq.(1.1) is derived by means of the averaging method from coupled dynamical systems having limit cycles, and now it is called the *Kuramoto model*.

It is obvious that when $K = 0$, $\theta_i(t)$ and $\theta_j(t)$ rotate on a circle at different velocities unless ω_i is equal to ω_j, and it is true for sufficiently small $K > 0$. On the other hand, if K is sufficiently large, it is numerically observed that some of oscillators or all of them tend to rotate at the same velocity on average, which is called the *synchronization* [18, 21]. If N is small, such a transition from de-synchronization to synchronization may be well revealed by means of the bifurcation theory [6, 13, 14]. However, if N is large, it is difficult to investigate the transition from the view point of the bifurcation theory and it is still far from understood.

[1] E mail address : chiba@math.kyushu-u.ac.jp

In order to evaluate whether synchronization occurs or not, Kuramoto introduced the *order parameter* $r(t)e^{\sqrt{-1}\psi(t)}$ by

$$r(t)e^{\sqrt{-1}\psi(t)} := \frac{1}{N}\sum_{j=1}^{N} e^{\sqrt{-1}\theta_j(t)}, \tag{1.2}$$

which gives the centroid of oscillators, where $r, \psi \in \mathbf{R}$. It seems that if synchronous state is formed, $r(t)$ takes a positive number, while if de-synchronization is stable, $r(t)$ is zero on time average.

The infinite-dimensional version or the continuous version of the Kuramoto model have been well investigated to reveal a bifurcation diagram of the order parameter (see [1, 3, 4, 5, 7, 15, 16, 21, 22, 23] and references therein). Such systems are rather tractable because the order parameter for the infinite-dimensional version can be constant in time, while the order parameter for the finite dimensional Kuramoto model is not constant in general because solutions fluctuate due to effects of finiteness [21]. Now the questions arise : How close is the order parameter of the infinite-dimensional version to that of the finite-dimensional Kuramoto model? What is the influence of finite size effects? This issue has been studied by many authors, see a reference paper [1] by Acebron *et al*. In particular, Daido [8] found the scaling law $|r - \langle r_N(t)\rangle| \sim (K_c - K)^{-1/2}N^{-1/2}$ for $K < K_c$ and $|r - \langle r_N(t)\rangle| \sim (K - K_c)^{-1/8}N^{-1/2}$ for $K > K_c$, although his analysis is not rigorous from a view point of mathematics, where r is the order parameter of the infinite-dimensional version in a steady state, $r_N(t)$ is the order parameter of the N-dimensional Kuramoto model, $\langle\ \rangle$ denotes the time average, and where K_c is the transition point from de-synchronization to synchronization.

In this paper, the continuous model, the continuous limit of the Kuramoto model, is introduced. It is proved that the order parameter of the N-dimensional Kuramoto model converges to that of the continuous model in the sense of probability, and their difference is of $O(N^{-1/2})$ as $N \to \infty$ for each t (note that we do not take time average). To prove this, the (m, k)-th moments are defined for both of the continuous model and the Kuramoto model. In particular, $(0, 1)$-th moment is the Kuramoto's order parameter. It is remarkable that both of the continuous model and the N-dimensional Kuramoto model become the same evolution equation, called the *moments system*, if they are rewritten by using the moments. It means that any solutions of the continuous model and the N-dimensional Kuramoto model for any N are embedded in the phase space of the moments system. This fact allows us to measure the distance between a solution of the continuous model and that of the N-dimensional Kuramoto model in the same phase space. These results and the central limit theorem prove that the difference between the order parameter of the N-dimensional Kuramoto model and that of the continuous model is of order $O(N^{-1/2})$ for each t, provided that initial values and natural frequencies for the N-dimensional Kuramoto model are independent and identically distributed according to a suitable probability measure.

More generally, a globally coupled phase oscillators defined to be

$$\frac{d\theta_i}{dt} = \omega_i + \frac{K}{N}\sum_{j=1}^{N} f(\theta_j - \theta_i),\ \ i = 1, \cdots, N, \tag{1.3}$$

is called the *Kuramoto-Daido model* [9, 7], where the 2π-periodic function $f : \mathbf{R} \to \mathbf{R}$ is called the *coupling function*. The results in this paper are easily extended to the Kuramoto-Daido model.

2 Continuous model

In this section, we introduce a continuous model of the Kuramoto model and show existence, uniqueness and other properties of solutions of the model used in a later section.

Let us consider the Kuramoto model (1.1). Following Kuramoto, we introduce the *order parameter* $\hat{Z}_1^0$ by

$$\hat{Z}_1^0(t) = \frac{1}{N}\sum_{j=1}^{N} e^{\sqrt{-1}\theta_j(t)}. \tag{2.1}$$

The quantities $\hat{Z}_k^m$ will be defined in the next section. By using it, Eq.(1.1) is rewritten as

$$\frac{d\theta_i}{dt} = \omega_i + \frac{K}{2\sqrt{-1}}(\hat{Z}_1^0(t)e^{-\sqrt{-1}\theta_i} - \overline{\hat{Z}_1^0(t)}e^{\sqrt{-1}\theta_i}), \tag{2.2}$$

where $\overline{\hat{Z}_1^0}$ denotes the complex conjugate of $\hat{Z}_1^0$. Motivated by these equations, we introduce a continuous model of the Kuramoto model, which is an evolution equation of a probability measure $\rho_t = \rho_t(\theta, \omega)$ on $S^1 \times \mathbf{R}$ parameterized by $t \in \mathbf{R}$, as

$$\left\{\begin{array}{l} \dfrac{\partial \rho_t}{\partial t} + \dfrac{\partial}{\partial \theta}\left(\left(\omega + \dfrac{K}{2\sqrt{-1}}(Z_1^0(t)e^{-\sqrt{-1}\theta} - \overline{Z_1^0(t)}e^{\sqrt{-1}\theta})\right)\rho_t\right) = 0, \\ Z_1^0(t) := \displaystyle\int_{\mathbf{R}}\int_0^{2\pi} e^{\sqrt{-1}\theta} d\rho_t, \\ \rho_0(\theta, \omega) = h(\theta, \omega), \end{array}\right. \tag{2.3}$$

where $h(\theta, \omega)$ is an initial measure. The $Z_1^0(t)$ is a continuous version of $\hat{Z}_1^0(t)$, and we also call it the *order parameter*. If we regard

$$v := \omega + \frac{K}{2\sqrt{-1}}(Z_1^0(t)e^{-\sqrt{-1}\theta} - \overline{Z_1^0(t)}e^{\sqrt{-1}\theta})$$

as a velocity field, Eq.(2.3) provides an equation of continuity $\partial\rho_t/\partial t + \partial(\rho_t v)/\partial\theta = 0$ known in fluid dynamics. It is easy to prove the low of conservation of mass:

$$\int_{\mathbf{R}}\int_0^{2\pi} \chi_E(\omega) d\rho_t = \int_{\mathbf{R}}\int_0^{2\pi} \chi_E(\omega) dh =: g(E), \tag{2.4}$$

where E is any Borel set on $\mathbf{R}$ and $\chi_E(\omega)$ is the characteristic function on E. A function g defined as above gives a probability measure for natural frequencies $\omega \in \mathbf{R}$ such that $\int_{\mathbf{R}} dg = 1$. In particular $\int_{\mathbf{R}}\int_0^{2\pi} d\rho_t = 1$ if $\int_{\mathbf{R}}\int_0^{2\pi} dh = 1$.

By using the characteristic curve method, Eq.(2.3) is formally integrated as follows: Consider the equation

$$\frac{dx}{dt} = \omega + \frac{K}{2\sqrt{-1}}(Z_1^0(t)e^{-\sqrt{-1}x} - \overline{Z_1^0(t)}e^{\sqrt{-1}x}), \ \ x \in S^1, \tag{2.5}$$

which defines a characteristic curve. Let $x = x(t, s; \theta, \omega)$ be a solution of Eq.(2.5) satisfying the initial condition $x(s, s; \theta, \omega) = \theta$. Then, along a characteristic curve, ρ_t is given as

$$\rho_t(\theta, \omega) = h(x(0, t; \theta, \omega), \omega) \exp\Big[\frac{K}{2}\int_0^t (Z_1^0(s)e^{-\sqrt{-1}x(s,t;\theta,\omega)} + \overline{Z_1^0(s)}e^{\sqrt{-1}x(s,t;\theta,\omega)})ds\Big]. \tag{2.6}$$

By using Eq.(2.6), it is easy to show the equality

$$\int_{\mathbf{R}}\int_0^{2\pi} a(\theta, \omega)d\rho_t = \int_{\mathbf{R}}\int_0^{2\pi} a(x(t, 0; \theta, \omega), \omega)dh, \tag{2.7}$$

for any measurable function $a(\theta, \omega)$. In particular, the order parameter $Z_1^0(t)$ are rewritten as

$$Z_1^0(t) = \int_{\mathbf{R}}\int_0^{2\pi} e^{\sqrt{-1}x(t,0;\theta,\omega)}dh. \tag{2.8}$$

Substituting it into Eqs.(2.5), (2.6), we obtain

$$\frac{d}{dt}x(t, s; \theta, \omega) = \omega + K\int_{\mathbf{R}}\int_0^{2\pi} \sin\Big(x(t, 0; \theta', \omega') - x(t, s; \theta, \omega)\Big)dh(\theta', \omega'), \tag{2.9}$$

and

$$\rho_t(\theta, \omega) = h(x(0, t; \theta, \omega), \omega) \exp\Big[K\int_0^t ds\cdot\int_{\mathbf{R}}\int_0^{2\pi} \cos\Big(x(s, 0; \theta', \omega') - x(s, t; \theta, \omega)\Big)dh(\theta', \omega')\Big]. \tag{2.10}$$

Even if $h(\theta, \omega)$ is not differentiable, we consider Eq.(2.10) to be a weak solution of Eq.(2.3). Indeed, even if h and ρ_t are not differentiable, the quantity (2.7) is differentiable with respect to t when $a(\theta, \omega)$ is differentiable.

Theorem 2.1. (i) There exists a unique weak solution ρ_t of the initial value problem (2.3) for any $t \geq 0$.

(ii) Solutions of (2.3) depend continuously on initial measures with respect to the weak topology in the sense that for any numbers $T, \varepsilon > 0$ and for any continuous function $a(\theta, \omega)$ on $S^1 \times \mathbf{R}$, there exist numbers $M(b) > 0$ and $\delta = \delta(T, \varepsilon, a) > 0$ such that if initial measures h_1, h_2 satisfy

$$\Big|\int_{\mathbf{R}}\int_0^{2\pi} b(\theta, \omega)(dh_1 - dh_2)\Big| < M(b)\delta, \tag{2.11}$$

for any continuous function $b(\theta, \omega)$, then solutions $\rho_{t,1}$ and $\rho_{t,2}$ with $\rho_{0,1} = h_1$ and $\rho_{0,2} = h_2$ satisfy

$$\left| \int_{\mathbf{R}} \int_0^{2\pi} a(\theta, \omega)(d\rho_{t,1} - d\rho_{t,2}) \right| < \varepsilon, \tag{2.12}$$

for $0 \leq t \leq T$. In particular if a is Lipschitz continuous, then $\varepsilon \sim O(\delta)$ as $\delta \to 0$.

Proof of (i). It is sufficient to prove that the integro-ODE (2.9) has a unique solution $x(t, s; \theta, \omega)$ satisfying $x(s, s; \theta, \omega) = \theta$ for any $t, s \geq 0$ and $\theta \in S^1$. Let us define a sequence $\{x_n(t, 0; \theta, \omega)\}_{n=0}^{\infty}$ to be

$$x_{n+1}(t, 0; \theta, \omega) = x_0(t, 0; \theta, \omega) + K \int_0^t d\tau \cdot \int_{\mathbf{R}} \int_0^{2\pi} f(x_n(\tau, 0; \theta', \omega') - x_n(\tau, 0; \theta, \omega)) dh(\theta', \omega') \tag{2.13}$$

and $x_0(t, 0; \theta, \omega) = \theta + \omega t$, where $f(\theta) = \sin\theta$ (since we prove the theorem for any C^1 function $f(\theta)$, the theorem is also true for the continuous model for the Kuramoto-Daido model (1.3)). We estimate $|x_{n+1}(t, 0; \theta, \omega) - x_n(t, 0; \theta, \omega)|$ as

$$\begin{aligned}
& |x_{n+1}(t, 0; \theta, \omega) - x_n(t, 0; \theta, \omega)| \\
\leq \;& K \int_0^t d\tau \cdot \int_{\mathbf{R}} \int_0^{2\pi} \Big| f(x_n(\tau, 0; \theta', \omega') - x_n(\tau, 0; \theta, \omega)) \\
& \qquad - f(x_{n-1}(\tau, 0; \theta', \omega') - x_{n-1}(\tau, 0; \theta, \omega)) \Big| dh(\theta', \omega') \\
\leq \;& KL \int_0^t d\tau \cdot \int_{\mathbf{R}} \int_0^{2\pi} \Big(|x_n(\tau, 0; \theta', \omega') - x_{n-1}(\tau, 0; \theta', \omega')| \\
& \qquad + |x_n(\tau, 0; \theta, \omega) - x_{n-1}(\tau, 0; \theta, \omega)| \Big) dh(\theta', \omega'),
\end{aligned}$$

where $L > 0$ is the Lipschitz constant of the function $f(\theta)$. When $n = 0$, we obtain

$$\begin{aligned}
|x_1(t, 0; \theta, \omega) - x_0(t, 0; \theta, \omega)| &\leq K \int_0^t d\tau \cdot \int_{\mathbf{R}} \int_0^{2\pi} | f(x_0(\tau, 0; \theta', \omega') - x_0(\tau, 0; \theta, \omega)) | \, dh(\theta', \omega') \\
&\leq KMt,
\end{aligned}$$

where $M = \max |f(\theta)|$. Thus we can show by induction that

$$|x_n(t, 0; \theta, \omega) - x_{n-1}(t, 0; \theta, \omega)| \leq 2^{n-1} L^{n-1} K^n M \frac{t^n}{n!}. \tag{2.14}$$

This proves that $x_n(t, 0; \theta, \omega)$ converges to a solution of the equation

$$\frac{dx}{dt}(t, 0; \theta, \omega) = \omega + K \int_{\mathbf{R}} \int_0^{2\pi} f(x(t, 0; \theta', \omega') - x(t, 0; \theta, \omega)) dh(\theta', \omega'),$$

as $n \to \infty$ for small $t \geq 0$. Existence of global solutions are easily obtained by a standard way because the phase space S^1 is compact; that is, solutions are extended for any $t > 0$.

Uniqueness of solutions is also proved in a standard way and the detail is omitted. With this $x(t,0;\theta,\omega)$, we define a sequence $\{x_n(t,s;\theta,\omega)\}_{n=0}^{\infty}$ to be

$$x_{n+1}(t,s;\theta,\omega) = x_0(t,s;\theta,\omega) + K\int_0^t d\tau\cdot\int_{\mathbf{R}}\int_0^{2\pi} f(x(\tau,0;\theta',\omega') - x_n(\tau,s;\theta,\omega))dh(\theta',\omega') \tag{2.15}$$

and $x_0(t,s;\theta,\omega) = \theta + \omega(t-s)$. Then, existence and uniqueness of global solutions $x(t,s;\theta,\omega)$ is proved in the same way as above. For this $x(t,s;\theta,\omega)$, Eq.(2.10) gives a (weak) solution of Eq.(2.3).

Proof of (ii). Suppose that initial measures h_1, h_2 satisfy Eq.(2.11). Let $\rho_{t,1}$ and $\rho_{t,2}$ be solutions of Eq.(2.3) satisfying $\rho_{0,1} = h_1$ and $\rho_{0,2} = h_2$. Let $x_i = x_i(t,0;\theta,\omega)$, $(i=1,2)$ be solutions of

$$\frac{dx_i}{dt} = \omega + K\int_{\mathbf{R}}\int_0^{2\pi} f(x_i(t,0;\theta',\omega') - x_i(t,0;\theta,\omega))dh_i(\theta',\omega'),\ \ x_i\in S^1, \tag{2.16}$$

satisfying $x_i(0,0;\theta,\omega) = \theta$, respectively. Then we obtain

$$\begin{aligned}
& \frac{d}{dt}\left(x_1(t,0;\theta,\omega) - x_2(t,0;\theta,\omega)\right) \\
= \ & K\int_{\mathbf{R}}\int_0^{2\pi} f(x_1(t,0;\theta',\omega') - x_1(t,0;\theta,\omega))dh_1(\theta',\omega') \\
& -K\int_{\mathbf{R}}\int_0^{2\pi} f(x_2(t,0;\theta',\omega') - x_2(t,0;\theta,\omega))dh_2(\theta',\omega') \\
= \ & K\int_{\mathbf{R}}\int_0^{2\pi} f(x_1(t,0;\theta',\omega') - x_1(t,0;\theta,\omega))(dh_1(\theta',\omega') - dh_2(\theta',\omega')) \\
& +K\int_{\mathbf{R}}\int_0^{2\pi}\Big(f(x_1(t,0;\theta',\omega') - x_1(t,0;\theta,\omega)) \\
& \qquad -f(x_2(t,0;\theta',\omega') - x_2(t,0;\theta,\omega))\Big)dh_2(\theta',\omega').
\end{aligned} \tag{2.17}$$

Integrating it yields

$$\begin{aligned}
& |x_1(t,0;\theta,\omega) - x_2(t,0;\theta,\omega)| \\
\leq \ & \int_0^t \Big| K\int_{\mathbf{R}}\int_0^{2\pi} f(x_1(s,0;\theta',\omega') - x_1(s,0;\theta,\omega))(dh_1(\theta',\omega') - dh_2(\theta',\omega'))\Big| ds \\
& + \int_0^t K\int_{\mathbf{R}}\int_0^{2\pi}\Big| f(x_1(s,0;\theta',\omega') - x_1(s,0;\theta,\omega)) \\
& \qquad -f(x_2(s,0;\theta',\omega') - x_2(s,0;\theta,\omega))\Big| dh_2(\theta',\omega')ds \\
\leq \ & KM\delta t + KL\int_0^t\int_{\mathbf{R}}\int_0^{2\pi}\Big(|x_1(s,0;\theta',\omega') - x_2(s,0;\theta',\omega')| \\
& \qquad +|x_1(s,0;\theta,\omega) - x_2(s,0;\theta,\omega)|\Big)dh_2(\theta',\omega')ds,
\end{aligned} \tag{2.18}$$

where L is the Lipschitz constant of f and M is a constant arising from Eq.(2.11). If we put

$$F(t,\theta,\omega) = |x_1(t,0;\theta,\omega) - x_2(t,0;\theta,\omega)|,$$

then (2.18) provides

$$\int_{\mathbf{R}}\int_0^{2\pi} F(t,\theta,\omega)dh_2 \leq KM\delta t + 2KL\int_0^t\int_{\mathbf{R}}\int_0^{2\pi} F(s,\theta,\omega)dh_2 ds.$$

Now the Gronwall inequality proves

$$\int_{\mathbf{R}}\int_0^{2\pi} F(t,\theta,\omega)dh_2 \leq \frac{M\delta}{2L}(e^{2KLt}-1).$$

Substituting it into (2.18) yields

$$\begin{aligned}
& |x_1(t,0;\theta,\omega) - x_2(t,0;\theta,\omega)| \\
\leq\ & KM\delta t + \frac{KM\delta}{2}\int_0^t (e^{2KLs}-1)ds + KL\int_0^t |x_1(s,0;\theta,\omega) - x_2(s,0;\theta,\omega)|ds \\
\leq\ & \frac{KM\delta t}{2} + \frac{M\delta}{4L}(e^{2KLt}-1) + KL\int_0^t |x_1(s,0;\theta,\omega) - x_2(s,0;\theta,\omega)|ds.
\end{aligned}$$

The Gronwall inequality is applied again to obtain

$$|x_1(t,0;\theta,\omega) - x_2(t,0;\theta,\omega)| \leq \frac{M\delta}{2L}(e^{2KLt}-1). \tag{2.19}$$

Finally, the left hand side of Eq.(2.12) is estimated as

$$\begin{aligned}
& \left|\int_{\mathbf{R}}\int_0^{2\pi} a(\theta,\omega)(d\rho_{t,1} - d\rho_{t,2})\right| \\
=\ & \left|\int_{\mathbf{R}}\int_0^{2\pi}\Big(a(x_1(t,0;\theta,\omega),\omega)dh_1 - a(x_2(t,0;\theta,\omega),\omega)dh_2\Big)\right| \\
\leq\ & \int_{\mathbf{R}}\int_0^{2\pi} |a(x_1(t,0;\theta,\omega),\omega) - a(x_2(t,0;\theta,\omega),\omega)|\, dh_2 \\
& \qquad + \left|\int_{\mathbf{R}}\int_0^{2\pi} a(x_1(t,0;\theta,\omega),\omega)(dh_1 - dh_2)\right|.
\end{aligned} \tag{2.20}$$

Since $a(\theta,\omega)$ is continuous and since Eq.(2.19) holds, the first term in the right hand side of the above is less than $\varepsilon/2$ for $0 \leq t \leq T$ if δ is sufficiently small. The second term is also less than $\varepsilon/2$ if δ is sufficiently small because of Eq.(2.11). This proves Eq.(2.12). It is easy to see by Eq.(2.20) that if $a(\theta,\omega)$ is Lipschitz continuous, then ε is of order $O(\delta)$. ■

3 Moments system

In this section, we introduce a moments system to transform the finite-dimensional Kuramoto model (1.1) and its continuous model (2.3) into the same system. We prove by using the moments system that the order parameter (1.2) for the Kuramoto model converges to the order parameter $Z_1^0(t)$ for the continuous model as $N \to \infty$ under appropriate assumptions.

For a given probability measure $h(\theta, \omega)$ on $S^1 \times \mathbf{R}$, suppose that absolute moments

$$M_k^n := \int_{\mathbf{R}} \int_0^{2\pi} |\omega^n e^{\sqrt{-1}k\theta}| dh \tag{3.1}$$

exist for $k = 0, \pm 1, \cdots$ and $m = 0, 1, \cdots$. Then, the moments m_k^n are defined to be

$$m_k^n := \int_{\mathbf{R}} \int_0^{2\pi} \omega^n e^{\sqrt{-1}k\theta} dh. \tag{3.2}$$

Conversely, if there exists a unique probability measure h for a given sequence of numbers $\{m_k^n\}_{n,k}$ such that Eq.(3.2) holds, then h is called *M-determinate*. In this case, we also say that moments $\{m_k^n\}_{n,k}$ is M-determinate. Many (necessary and) sufficient conditions for which h is M-determinate have been well studied as the moment problem [2, 19, 10, 20]. For example, one of the most convenient conditions is that if h has all absolute moments M_k^n and they satisfy $\sum_{n=1}^{\infty} (M_0^n + 1)^{-1/n} = \infty$ (*Carleman's condition*), then h is M-determinate.

In what follows, we suppose that an initial measure $h(\theta, \omega)$ for the initial value problem (2.3) has all absolute moments and is M-determinate.

Example 3.1. If h has compact support, then h is M-determinate. Suppose that h has a probability density function of the form $\hat{h}(\theta)\hat{g}(\omega)$. If $\hat{g}(\omega)$ is the Gaussian distribution, then h is M-determinate. If $\hat{g}(\omega) = 1/(\pi(1 + \omega^2))$ is the Lorentzian distribution, h is *not* M-determinate because h does not have all moments.

Recall that a probability measure g for the natural frequency ω is defined through Eq.(2.4). Since h has absolute moments

$$M_k^n = \int_{\mathbf{R}} \int_0^{2\pi} |\omega^n e^{\sqrt{-1}k\theta}| dh = \int_{\mathbf{R}} |\omega|^n dg < \infty, \tag{3.3}$$

g also has all moments $\mu_n := \int_{\mathbf{R}} \omega^n dg$, $n = 0, 1, 2, \cdots$. Consider the Lebesgue space $L^2(\mathbf{R}, dg)$. Since all moments μ_m of g exist, we can construct a complete orthonormal system $\{P_m(\omega)\}_{m=0}^{\infty}$ on $L^2(\mathbf{R}, dg)$, by using the Gram-Schmidt orthogonalization from $\{\omega^m\}_{m=0}^{\infty}$, such that

$$(P_n, P_m) = \int_{\mathbf{R}} P_n(\omega) P_m(\omega) dg = \begin{cases} 1 & (n = m) \\ 0 & (n \neq m), \end{cases} \tag{3.4}$$

where $(\ ,\ )$ denotes the inner product on $L^2(\mathbf{R}, dg)$ and $P_n(\omega)$ is a polynomial of degree n. In particular, $P_0(\omega) \equiv 1$. It is well known that $P_n(\omega)$ satisfies the relation

$$\omega P_n(\omega) = b_n P_{n+1}(\omega) + a_n P_n(\omega) + b_{n-1} P_{n-1}(\omega) \tag{3.5}$$

for $n = 0, 1, 2 \cdots$, where a_n and b_n are real constants determined by g. The matrix $\mathcal{M}$ defined as

$$\mathcal{M} = \begin{pmatrix} a_0 & b_0 & & & \\ b_0 & a_1 & b_1 & & \\ & b_1 & a_2 & b_2 & \\ & & & \ddots & \\ & & & & \end{pmatrix} \tag{3.6}$$

is called the *Jacobi matrix* for g. Eq.(3.5) shows that the Jacobi matrix gives the $l^2(\mathbf{Z}_{\geq 0})$ representation of the multiplication operator

$$\mathcal{M} : p(\omega) \mapsto \omega p(\omega) \tag{3.7}$$

on $L^2(\mathbf{R}, dg)$, where $l^2(\mathbf{Z}_{\geq 0}) = \{\{x_n\}_{n=0}^{\infty} \mid \sum_{n=0}^{\infty} |x_n|^2 < \infty\}$.

If an initial measure h is M-determinate, so is a solution ρ_t of the continuous model (2.3) because of Eq.(2.10). Let us define the (m, k)*-th moments* Z_k^m for ρ_t to be

$$Z_k^m(t) = \int_{\mathbf{R}} \int_0^{2\pi} P_m(\omega) e^{\sqrt{-1}k\theta} d\rho_t, \tag{3.8}$$

for $m = 0, 1, 2 \cdots$ and $k = 0, \pm 1, \pm 2, \cdots$. In particular $Z_1^0(t)$ is the order parameter given in Eq.(2.3), and $Z_{-k}^m(t) = \overline{Z_k^m(t)}$. Note that

$$Z_0^m(t) = \begin{cases} 1 & (m = 0) \\ 0 & (m \neq 0) \end{cases} \tag{3.9}$$

are constants. It is easy to verify that

$$|Z_k^m(t)| \leq 1 \tag{3.10}$$

by using the Schwarz inequality. By using Eq.(2.7), an evolution equation for $Z_k^m(t)$ is

derived as

$$
\begin{aligned}
\frac{dZ_k^m}{dt} &= \frac{\partial}{\partial t}\int_{\mathbf{R}}\int_0^{2\pi} P_m(\omega)e^{\sqrt{-1}kx(t,0;\theta,\omega)}dh \\
&= \int_{\mathbf{R}}\int_0^{2\pi} P_m(\omega)\sqrt{-1}k\frac{\partial x}{\partial t}(t,0;\theta,\omega)e^{\sqrt{-1}kx(t,0;\theta,\omega)}dh \\
&= \int_{\mathbf{R}}\int_0^{2\pi} P_m(\omega)\sqrt{-1}k\left(\omega+\frac{K}{2\sqrt{-1}}(Z_1^0(t)e^{-\sqrt{-1}x(t,0;\theta,\omega)}-Z_{-1}^0(t)e^{\sqrt{-1}x(t,0;\theta,\omega)})\right)e^{\sqrt{-1}kx(t,0;\theta,\omega)}dh \\
&= \sqrt{-1}k\int_{\mathbf{R}}\int_0^{2\pi}\omega P_m(\omega)e^{\sqrt{-1}kx(t,0;\theta,\omega)}dh \\
&\quad +\frac{kK}{2}\int_{\mathbf{R}}\int_0^{2\pi} P_m(\omega)(Z_1^0(t)e^{\sqrt{-1}(k-1)x(t,0;\theta,\omega)}-Z_{-1}^0(t)e^{\sqrt{-1}(k+1)x(t,0;\theta,\omega)})dh \\
&= \sqrt{-1}k\int_{\mathbf{R}}\int_0^{2\pi}\left(b_mP_{m+1}(\omega)+a_mP_m(\omega)+b_{m-1}P_{m-1}(\omega)\right)e^{\sqrt{-1}k\theta}d\rho_t \\
&\quad +\frac{kK}{2}\int_{\mathbf{R}}\int_0^{2\pi} P_m(\omega)(Z_1^0(t)e^{\sqrt{-1}(k-1)\theta}-Z_{-1}^0(t)e^{\sqrt{-1}(k+1)\theta})d\rho_t \\
&= \sqrt{-1}k\left(b_mZ_k^{m+1}+a_mZ_k^m+b_{m-1}Z_k^{m-1}\right)+\frac{kK}{2}(Z_1^0Z_{k-1}^m-Z_{-1}^0Z_{k+1}^m).
\end{aligned}
\tag{3.11}
$$

Put $Z_k = (Z_k^0, Z_k^1, Z_k^2, \cdots)^T$, where T denotes the transpose. Define the Jacobi matrix $\mathcal{M}$ and the projection matrix $\mathcal{P}$ to be Eq.(3.6) and

$$
\mathcal{P} = \begin{pmatrix} 1 & 0 & \cdots & \\ 0 & 0 & & \\ \vdots & & \ddots & \\ & & & \end{pmatrix}, \tag{3.12}
$$

respectively. Then, Eq.(3.11) is rewritten as

$$
\frac{d}{dt}\begin{pmatrix} Z_1 \\ Z_2 \\ Z_3 \\ \vdots \end{pmatrix} = \begin{pmatrix} \sqrt{-1}\mathcal{M}+\frac{K}{2}\mathcal{P} & & & \\ & 2\sqrt{-1}\mathcal{M} & & \\ & & 3\sqrt{-1}\mathcal{M} & \\ & & & \ddots \end{pmatrix}\begin{pmatrix} Z_1 \\ Z_2 \\ Z_3 \\ \vdots \end{pmatrix} + \frac{K}{2}\begin{pmatrix} -Z_{-1}^0 Z_2 \\ 2(Z_1^0 Z_1 - Z_{-1}^0 Z_3) \\ 3(Z_1^0 Z_2 - Z_{-1}^0 Z_4) \\ \vdots \end{pmatrix}. \tag{3.13}
$$

Note that equations for $Z_{-1}, Z_{-2}, \cdots$ are omitted because $Z_{-k} = \overline{Z_k}$. The first term is a linear term and the second is a nonlinear term. We call Eq.(3.11) or Eq.(3.13) the *moments system*. The dynamics of the system will be investigated in forthcoming papers [4, 5].

Let MD be the set of M-determinate sequences $\{Z_k^m\}_{m,k}$ in the sense that if $\{Z_k^m\}_{m,k} \in MD$, then there exists a unique measure h on $S^1 \times \mathbf{R}$ such that $Z_k^m = \int_{\mathbf{R}}\int_0^{2\pi} P_m(\omega)e^{\sqrt{-1}k\theta}dh$. Since there is a one-to-one correspondence between elements of MD and M-determinate measures, Thm.2.1 is restated as follows.

Theorem 3.2. (i) There exists a unique solution $\{Z_k^m(t)\}_{m,k} \in MD$ of the moments system if an initial condition $\{Z_k^m(0)\}_{m,k}$ is in MD.
(ii) Let $\{Z_k^m(t)\}$ and $\{\widetilde{Z}_k^m(t)\}$ be solutions of the moments system with initial conditions $\{Z_k^m(0)\}$, $\{\widetilde{Z}_k^m(0)\} \in MD$, respectively. For any positive numbers T and ε, there exist positive numbers $C_{m,k}$ and $\delta = \delta(T, \varepsilon)$ such that if

$$|Z_k^m(0) - \widetilde{Z}_k^m(0)| < C_{m,k}\delta, \tag{3.14}$$

for any m, k, then the inequality

$$|Z_k^m(t) - \widetilde{Z}_k^m(t)| < \varepsilon \tag{3.15}$$

holds for $0 \leq t \leq T$. In particular $\varepsilon \sim O(\delta)$ as $\delta \to 0$.

For the N-dimensional Kuramoto model (1.1), we define the (m, k)-*th moments* to be

$$\hat{Z}_k^m(t) := \frac{1}{N}\sum_{j=1}^{N} P_m(\omega_j) e^{\sqrt{-1}k\theta_j(t)}, \tag{3.16}$$

for $m = 0, 1, 2, \cdots$ and $k = 0, \pm 1, \pm 2, \cdots$. In particular $\hat{Z}_1^0$ is the order parameter defined in Eq.(2.1). By using Eq.(1.1), it is easy to verify that $\hat{Z}_k^m(t)$'s satisfy a system of differential equations

$$\frac{d\hat{Z}_k^m(t)}{dt} = \sqrt{-1}k\left(b_m\hat{Z}_k^{m+1} + a_m\hat{Z}_k^m + b_{m-1}\hat{Z}_k^{m-1}\right) + \frac{kK}{2}(\hat{Z}_1^0\hat{Z}_{k-1}^m - \hat{Z}_{-1}^0\hat{Z}_{k+1}^m). \tag{3.17}$$

It is remarkable that Eq.(3.17) has the same form as Eq.(3.11). This means that all solutions of the Kuramoto model for any N are embedded in the phase space of the moments system (3.11). This fact allows us to prove Theorem 3.3 below. Originally the moments system for the Kuramoto model was introduced by Perez and Ritort [17], although their definition of the moments is $H_k^m := 1/N \cdot \sum_{j=1}^N \omega_j^m e^{\sqrt{-1}k\theta_j(t)}$. Since we adopt orthogonal polynomials $\{P_m(\omega)\}_{m=0}^\infty$ to define moments (3.16), our moments system is more suitable for mathematical analysis [4, 5].

Now we are in a position to show the main theorems in this paper, which states that differences between moments $Z_k^m(t)$ and $\hat{Z}_k^m(t)$ are of $O(1/\sqrt{N})$ and thus the continuous model (2.3) is proper to investigate the Kuramoto model (1.1) for large N.

Theorem 3.3. Let ρ_t be a solution of the continuous model (2.3) such that an initial measure $h(\theta, \omega)$ is M-determinate. Suppose that for the N-dimensional Kuramoto model (1.1), pairs $(\theta_j(0), \omega_j)$ of initial values $\theta_j(0)$, $j = 1, \cdots, N$ and natural frequencies ω_j, $j = 1, \cdots, N$ are independent and identically distributed according to the probability measure $h(\theta, \omega)$. Then, moments $Z_k^m(t)$ and $\hat{Z}_k^m(t)$ defined by Eqs.(3.8) and (3.16) satisfy

$$|Z_k^m(t) - \hat{Z}_k^m(t)| \to 0, \quad a.s.\ (N \to \infty), \tag{3.18}$$

for any m, k and t ($a.s.$ denotes " almost surely"). Further, for any positive number δ, there exists a number $C = C(m, k, t, \delta) > 0$ such that

$$P(\,|Z_k^m(t) - \hat{Z}_k^m(t)| < C/\sqrt{N}\,) \to 1 - \delta, \;\; (N \to \infty), \tag{3.19}$$

where $P(A)$ is the probability that an event A will occur.

Proof. The average of $\hat{Z}_k^m(0)$ is calculated as

$$\begin{aligned} E[\hat{Z}_k^m(0)] &= E[\frac{1}{N}\sum_{j=1}^{N} P_m(\omega_j)e^{\sqrt{-1}k\theta_j(0)}] \\ &= E[P_m(\omega_j)e^{\sqrt{-1}k\theta_j(0)}] \\ &= \int_{\mathbf{R}}\int_0^{2\pi} P_m(\omega)e^{\sqrt{-1}k\theta}dh = Z_k^m(0). \end{aligned} \tag{3.20}$$

Thus Eqs.(3.18) and (3.19) for $t = 0$ immediately follow from the strong law of large number and the central limit theorem, respectively. Note that the strong law of large number and the central limit theorem are no longer applicable for $t > 0$ because $\theta_j(t)$'s are not independent and identically distributed when t is positive. However, since $Z_k^m(t)$ and $\hat{Z}_k^m(t)$ satisfy the same moments system, and since solutions of the moments system are continuous with respect to initial values (Thm.3.2 (ii)), Eqs.(3.18),(3.19) hold for each positive t if they are true for $t = 0$. ■

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